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COLLISION TERMS FROM FLUCTUATIONS IN THE HTL THEORY FOR THE QUARK-GLUON PLASMA

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Abstract

Starting from the kinetic formulation of the hard thermal loop effective theory, we have (re)derived the collision terms for soft modes of order $g^2 T \log(1/g)$ by averaging the statistical fluctuations in the plasma.

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Over the past year, significant progress has been achieved in understanding the effective dynamics of soft excitations $p \sim g^2T$ in the non abelian quark-gluon plasma at high temperature. Bödeker [1] has derived a diffusive field theory that involves only gauge fields with dynamics governed by a Langevin equation, in which the gaussian noise is only parametrised by the color conductivity, which shows a logarithmic dependence on g. As an application, the numerical value of the leading log behavior of the sphaleron rate, which measures baryon number violation in the hot standard model, has been computed [2].

The crucial point in Bödeker's original derivation is the way in which the integration of hard $\sim T$ and semi-hard $\sim gT$ scales is done. The contribution of hard fields is easy to find. It is encoded in the local formulation of the hard-thermal-loop effective action [3,4] leading to a closed set of collisionless kinetic equations and field equations for the semi-hard modes. The remaining integration of the semi-hard scales turns out to be a difficult task. Bödecker's main result is an effective kinetic equation with collision terms arising from the averaging over semi-hard fields. This equation has also been proposed by Arnold, Son and Yaffe [5] by analyzing the scattering processes between hard particles in the plasma. Very recently, a Boltzmann equation has been rigorously derived by Blaizot and Iancu [6], starting from the Dyson-Schwinger equations. For the QCD plasma, it coincides with the one previously obtained in Refs. [1,5]

The main purpose of this paper is to give an alternative derivation *à la* Balescu-Lenard [7] of the collision terms for the longwavelength color deviations of equilibrium. The starting point is the set of "microscopic" dynamical equations coming from the hard-thermal loop effective action describing the evolution of the collisionless plasma. In this way, collision terms arise as statistical averages of correlators of the plasma fluctuations at equilibrium [8,9]. A similar derivation starting from classical transport theory has been given recently by Litim and Manuel [10]. It should be emphasized that the main point here, is to provide a derivation directly from the HTL effective action. As shown in Ref. [10], a strictly classical derivation leads to a different value for the color conductivity because the value of the Debye mass computed from a Maxwell-Boltzmann distribution is different.

Let us outline our calculations. Our construction relies on the local formulation of the HTL theory [3], which involves a set of coupled equations of motion for the mean fields and their induced current. The mean fields $A^a_\mu(x)$ satisfy the Yang-Mills equations with an induced current on the right hand side:

$$[D_\nu, F^{\nu\mu}]_a = j^\mu_a \, , \tag{1}$$

where $D_\nu O(x) = \partial_\nu O(x) + ig[A_\nu, O(x)]$, $A_\nu = A^a_\nu t^a$ and $F_{\mu\nu} = [D_\mu, D_\nu]/(ig)$. (The generators of the $SU(N_c)$ gauge group in the fundamental representation are denoted by t^a; they satisfy $[t^a, t^b] = if^{abc}t^c$ and $\mathrm{tr}(t^a t^b) = \delta^{ab}/2$.) Furthermore, the induced color current j^μ is related to the color fluctuations in the gluon color densities $\delta N(\mathbf{p}, x) = \delta N^a(\mathbf{p}, x)t^a$ and the quark color densities $\delta n_\pm(\mathbf{p}, x) = \delta n^a_\pm(\mathbf{p}, x)t^a$ by

$$j^\mu(x) = g \int \frac{d^3p}{(2\pi)^3} v^\mu \, [2N_c \delta N(\mathbf{p}, x) + N_f \delta n_+(\mathbf{p}, x) - N_f \delta n_-(\mathbf{p}, x)] \, . \tag{2}$$

In this equation, $v^\mu \equiv (1, \mathbf{v})$, $\mathbf{v} \equiv \mathbf{p}/p$ and N_f is the number of quark flavors. The system is closed by the non-abelian generalization of the Vlasov equations:

$$[v \cdot D, \delta N(\mathbf{p}, x)] = -g \, \mathbf{v} \cdot \mathbf{E}(x) \frac{\mathrm{d}N(p)}{\mathrm{d}p} \, , \tag{3}$$

$$[v \cdot D, \delta n_\pm(\mathbf{p}, x)] = \mp g \, \mathbf{v} \cdot \mathbf{E}(x) \frac{\mathrm{d}n(p)}{\mathrm{d}p} \, , \tag{4}$$

where $N(p) = 1/(\exp(\beta p) - 1)$, $n(p) = 1/(\exp(\beta p) + 1)$ and $E^i_a \equiv F^{i0}_a$. The current j^μ is covariantly conserved.

These equations contain the semi-hard degrees of freedom. In order to obtain a kinetic description for the soft modes, we decompose the fields into mean values and fluctuations:

$$A^\mu_a \rightarrow \overline{A}^\mu_a + \delta A^\mu_a \, , \tag{5}$$

$$\delta N \rightarrow \overline{\delta N} + \delta N \, , \tag{6}$$

$$\delta n_\pm \rightarrow \overline{\delta n}_\pm + \delta n_\pm \, . \tag{7}$$

and average the previous microscopic equations. As a result, one obtains a set of equations for the average soft gauge fields $\overline{A}^a_\mu(x)$, and for the mean values of the distribution functions $\overline{\delta N}$, $\overline{\delta n}_\pm$:

$$\left[\overline{D}_\nu, \overline{F}^{\nu\mu}\right]_a = \overline{j}_a^\mu + \text{non-linear terms}, \tag{8}$$

$$\left[v \cdot \overline{D}, \overline{\delta N}(\mathbf{p}, x)\right] + g\, \mathbf{v} \cdot < \mathbf{E}(x) > \frac{\mathrm{d}N(p)}{\mathrm{d}p} = -ig\, v \cdot < [\delta A, \delta N] >, \tag{9}$$

$$\left[v \cdot \overline{D}, \overline{\delta n}_\pm(\mathbf{p}, x)\right] \pm g\, \mathbf{v} \cdot < \mathbf{E}(x) > \frac{\mathrm{d}n(p)}{\mathrm{d}p} = -ig\, v \cdot < [\delta A, \delta n_\pm] >, \tag{10}$$

where $\overline{D}_\nu = \partial_\nu + ig[\overline{A}_\nu, \cdot]$ and $< \mathbf{E}^i >= \overline{F}^{i0}$. Thus, the set of equations above includes the "collision integrals" which are determined by the correlators of the fluctuations. A noteworthy feature of the collision terms is their non-Abelian origin. A similar analysis starting with the abelian Vlasov-Maxwell equations for a QED plasma leads to no collision terms because of the linearity underlying this case.

The main difficulty in computing the collision terms is caused by the nonlinearity of the equations for the fluctuations. However, the situation becomes significantly simpler for a plasma at high temperature with the coupling constant assumed to be small. In this case, we can neglect terms of higher order in the fluctuations [8] and the following linearized equations for the fluctuations can be used:

$$\left[\overline{D}_\nu, \delta F^{\nu\mu}\right]_a + ig\left[\delta A_\nu, \overline{F}^{\nu\mu}\right]_a = \delta j_a^\mu, \tag{11}$$

$$\left[v \cdot \overline{D}, \delta N(\mathbf{p}, x)\right] + ig\, v \cdot \left[\delta A, \overline{\delta N}\right] = -g\, \mathbf{v} \cdot \delta\mathbf{E}(x)\frac{\mathrm{d}N(p)}{\mathrm{d}p}, \tag{12}$$

$$\left[v \cdot \overline{D}, \delta n_\pm(\mathbf{p}, x)\right] + ig\, v \cdot \left[\delta A, \overline{\delta n}_\pm\right] = \mp g\, \mathbf{v} \cdot \delta\mathbf{E}(x)\frac{\mathrm{d}n(p)}{\mathrm{d}p}, \tag{13}$$

where $\delta F_{\mu\nu} = \overline{D}_\mu \delta A_\nu - \overline{D}_\nu \delta A_\mu$ and $\delta E^i = \delta F^{i0}$. In order to obtain the explicit form of the kinetic equations we are looking for, it will suffice to linearize the collision terms in the mean values of the distribution functions, since the left-hand sides of Eqs. (9)–(10) are already linear functions of them. Thus, the task is to compute the fluctuations up to first order in $\overline{\delta N}$, $\overline{\delta n}_\pm$ and zero order in $\overline{A}$. Moreover, I will only consider mean color distributions with no dependence in $(t, \mathbf{x})$, which means that the results to be derived must be considered as zero-order terms in some derivative expansion.

In order to implement the suggested programme, it is customary to use Laplace transforms, such as

$$\delta N_a(\omega,\mathbf{k};\mathbf{p}) = \int d^3x \int_0^\infty dt\, e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \delta N_a(t,\mathbf{x};\mathbf{p}) \,, \tag{14}$$

$$\delta A_a^i(\omega,\mathbf{k}) = \int d^3x \int_0^\infty dt\, e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \delta A_a^i(t,\mathbf{x}) \,, \tag{15}$$

In terms of these, when the Coulomb gauge is used, the first order correlator can be written as

$$\begin{aligned} < \delta A_a^i(\omega,\mathbf{k})\delta N_b(\omega',\mathbf{k}';\mathbf{p}) >^{(1)} &= \frac{ig f^{bmn} \overline{\delta N}^n(\mathbf{p}) \hat{p}^j}{\omega - \mathbf{k}\cdot\hat{\mathbf{p}}} < \delta A_a^i(\omega,\mathbf{k}) \delta A_m^j(\omega',\mathbf{k}') > \\ &- \frac{ig^2 f^{amn}}{k^2 - \omega^2 \epsilon_\perp(\omega,k)} < \delta A_m^k(\omega,\mathbf{k}) \delta N_b(\omega',\mathbf{k}';\mathbf{p}) > \left(\delta^{ij} - \hat{k}^i \hat{k}^j\right) \\ &\times \int \frac{d^3p'}{(2\pi)^3} \frac{\hat{p}'^j \hat{p}'^k}{\omega - \mathbf{k}\cdot\hat{\mathbf{p}}'} \left(2N_c \overline{\delta N}^n(\mathbf{p}') + N_f \overline{\delta n}_+^n(\mathbf{p}') - N_f \overline{\delta n}_-^n(\mathbf{p}')\right) \,, \end{aligned} \tag{16}$$

with a similar expression for $< \delta A_a^i(\omega,\mathbf{k}) \delta n_\pm^b(\omega',\mathbf{k}';\mathbf{p}) >^{(1)}$. In this equation, $\epsilon_\perp(\omega,k)$ represents the transverse part of the dielectric function, defined by

$$\begin{aligned} \epsilon^{ij}(\omega,k) &= \epsilon_\parallel(\omega,k) \hat{k}^i \hat{k}^j + \epsilon_\perp(\omega,k)(\delta^{ij} - \hat{k}^i \hat{k}^j) \\ &= \delta^{ij} + \frac{2g^2}{\omega} \int \frac{d^3p}{(2\pi)^3} \frac{\hat{p}^i \hat{p}^j}{\omega - \mathbf{k}\cdot\hat{\mathbf{p}}} \left(N_c N'(p) + N_f\, n'(p)\right) \,. \end{aligned} \tag{17}$$

Eqs. (16) and (17) deserve some comments:

i) The prescriptions $\omega \to \omega + i0^+$, $\omega' \to \omega' + i0^+$ are implicit in all denominators because the (double) Mellin's inversion formula for Laplace transforms requires that all singularities lie in the lower half ω, ω' planes.

ii) the correlators over the initial conditions for $t = 0$,

$$< \delta N_a(\mathbf{k};\mathbf{p}) \delta N_b(\mathbf{k}';\mathbf{p}') > \; = (2\pi)^3 \delta(\mathbf{k}+\mathbf{k}') \delta^{ab} \left((2\pi)^3 \delta(\mathbf{p}-\mathbf{p}') f_N(p) + \mu_N(\mathbf{k};\mathbf{p},\mathbf{p}')\right) \,, \tag{18}$$

$$< \delta n_\pm^a(\mathbf{k};\mathbf{p}) \delta n_\pm^b(\mathbf{k}';\mathbf{p}') > \; = (2\pi)^3 \delta(\mathbf{k}+\mathbf{k}') \delta^{ab} \left((2\pi)^3 \delta(\mathbf{p}-\mathbf{p}') f_n(p) + \mu_n(\mathbf{k};\mathbf{p},\mathbf{p}')\right) \,, \tag{19}$$

$$< \delta A_a^i(\mathbf{k}) \delta A_b^j(\mathbf{k}') > \; = (2\pi)^3 \delta(\mathbf{k}+\mathbf{k}') \delta^{ab} \left(\delta^{ij} - \hat{k}^i \hat{k}^j\right) \mu_A(\mathbf{k}) \,, \tag{20}$$

have played an important role, since in Eqs. (16) and (17) only potential non-zero contributions to the large time behavior have been written. This is so because, as proven in [9], when $f_{N,n}(p)$, $\mu_{N,n}(\mathbf{k};\mathbf{p},\mathbf{p}'))$ and $\mu_A(\mathbf{k})$ are smooth functions of the momenta, the only contributions giving a non damped function of t and t' come from the functions $f_{N,n}(p)$. Thus, the

initial conditions for δA, $\mathrm{d}\delta A/\mathrm{d}t$ and the specific form of the two-particle correlation functions do not matter. All the information about the large time behavior is included in $f_{N,n}(p)$ whose explicit form can be determined as follows. After substitution of Eqs. (18)–(20) into the appropriate correlators we find

$$
\begin{aligned}
< \delta A_a^i \ (\ \omega, \mathbf{k})\delta A_b^j(\omega', \mathbf{k}') > &\equiv (2\pi)^4 \delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega')\delta^{ab}\left(\delta^{ij} - \hat{k}^i\hat{k}^j\right) S(\omega, k) \\
&= (2\pi)^4 \delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega')\delta^{ab}\left(\delta^{ij} - \hat{k}^i\hat{k}^j\right) \qquad (21) \\
&\times \frac{g^2(1-\omega^2/k^2)}{|-\omega^2\epsilon_\perp(\omega,k)+k^2|^2}\,\frac{1}{4\pi k}\int_0^\infty dp\, p^2 \left(4N_c^2 f_N(p) + 2N_f^2 f_n(p)\right) . \qquad (22)
\end{aligned}
$$

Then, a comparison with the prediction of the fluctuation-dissipation theorem relating the correlator of field fluctuations with the imaginary part of the retarded propagator,

$$
S(\omega, k) = (1 - e^{-\beta\omega})^{-1} 2\,\mathrm{Im}\,\frac{1}{k^2 - (\omega + i0^+)^2 \epsilon_\perp(\omega + i0^+, k)} , \tag{23}
$$

reveals the right values for the initial correlators,

$$
f_N(p) = N_c^{-1}\, N(p)(1+N(p)) = -N_c^{-1} T N'(p) , \tag{24}
$$

$$
f_n(p) = 2N_f^{-1} n(p)(1-n(p)) = -2N_f^{-1} T n'(p) , \tag{25}
$$

where the factors N_c^{-1} and $2N_f^{-1}$ compensate similar factors in the color current. These results for the initial correlators could also be established from the general properties of the fluctuations of an ideal gas [11]. Up to the color and flavor factors, Eqs. (24) and (25) are exactly the equations for the mean square of the fluctuations presented in Ref. [11][1].

Now, following the Ref. [9], other correlators can be easily computed. For example, at leading order in g we have

$$
\begin{aligned}
< \delta A_a^i(\omega, \mathbf{k}) \delta N_b(\omega', \mathbf{k}'; \mathbf{p}) > \ &= (2\pi)^4 \delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega')\, 4\pi g\, \delta^{ab}\left(\delta^{ij} - \hat{k}^i\hat{k}^j\right)\hat{p}^j \\
&\times \frac{N(p)(1+N(p))}{k^2 - \omega^2\epsilon_\perp(\omega,k)}\,\delta(\omega - \mathbf{k}\cdot\hat{\mathbf{p}}) , \qquad (26) \\
< \delta A_a^i(\omega, \mathbf{k}) \delta n_\pm^b(\omega', \mathbf{k}'; \mathbf{p}) > \ &= \pm(2\pi)^4 \delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega')\, 2\pi g\, \delta^{ab}\left(\delta^{ij} - \hat{k}^i\hat{k}^j\right)\hat{p}^j \\
&\times \frac{n(p)(1-n(p))}{k^2 - \omega^2\epsilon_\perp(\omega,k)}\,\delta(\omega - \mathbf{k}\cdot\hat{\mathbf{p}}) . \qquad (27)
\end{aligned}
$$

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[1]Specifically, Eqs. (113,3) and (113,4) of Ref. [11].

iii) The dominant scattering processes in the collisions correspond to t-channel exchange of quasistatic magnetic gauge bosons, which means that the longitudinal part of the interaction can be ignored. Also, as the relevant regime to be considered in transverse propagators is $\omega \ll k$, we can make the following replacements

$$\epsilon_\perp(\omega,k) \simeq \frac{3\pi\, i\, \omega_p^2}{4\omega k}\,, \tag{28}$$

$$S(\omega,k) \simeq \frac{2\pi}{k^2\beta}\,\delta(\omega)\,, \tag{29}$$

$$\frac{1}{(-k^2+\omega^2\epsilon_\perp(\omega,k))^2} \simeq \frac{2}{3k\omega_p^2}\,\delta(\omega)\,, \tag{30}$$

with the plasma frequency $\omega_p^2 = g^2(2N_c+N_f)T^2/18$.

Finally, the collision terms easily emerge from all of this by integration over $(\omega,\mathbf{k})$. Their explicit form turns out to be

$$\begin{aligned} -\,ig\,v\cdot <[\delta A,\delta N]^a> = -\gamma_g \Bigg[\overline{\delta N}^a(x,\mathbf{p}) + \frac{4g^2}{3\pi\omega_p^2}\,N'(p) \\ \times \int \frac{d^3p'}{(2\pi)^3}\frac{(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')^2}{\sqrt{1-(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')^2}}\left(2N_c\overline{\delta N}^a(x,\mathbf{p}') + N_f\overline{\delta n}^a_+(x,\mathbf{p}') - N_f\overline{\delta n}^a_-(x,\mathbf{p}')\right)\Bigg]\,, \end{aligned} \tag{31}$$

$$\begin{aligned} -ig\,v\cdot <[\delta A,\delta n_\pm]^a> = -\gamma_g \Bigg[\overline{\delta n}^a_\pm(x,\mathbf{p}) \pm \frac{4g^2}{3\pi\omega_p^2}\,n'(p) \\ \times \int \frac{d^3p'}{(2\pi)^3}\frac{(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')^2}{\sqrt{1-(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')^2}}\left(2N_c\overline{\delta N}^a(x,\mathbf{p}') + N_f\overline{\delta n}^a_+(x,\mathbf{p}') - N_f\overline{\delta n}^a_-(x,\mathbf{p}')\right)\Bigg]\,, \end{aligned} \tag{32}$$

with the damping rate γ_g for the hard thermal gauge bosons

$$\gamma_g = \frac{g^2N_cT}{4\pi}\log(1/g)\,. \tag{33}$$

If we combine the different color distributions functions in the form [3,5]

$$W^a(x,\hat{\mathbf{p}}) = \frac{1}{3\omega_p^2}\int\frac{dp\,4\pi p^2}{(2\pi)^3}\left(2N_c\overline{\delta N}^a(x,\mathbf{p}) + N_f\overline{\delta n}^a_+(x,\mathbf{p}) - N_f\overline{\delta n}^a_-(x,\mathbf{p})\right)\,, \tag{34}$$

we obtain

$$\left[v\cdot\overline{D}, W(x,\hat{\mathbf{p}})\right]^a - \mathbf{v}\cdot <\mathbf{E}^a(x)> = -\delta C[W]^a\,, \tag{35}$$

$$\delta C[W]^a = \gamma_g\left(W^a(x,\hat{\mathbf{p}}) - \frac{4}{\pi}\int d\Omega_{p'}\frac{(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')^2}{\sqrt{1-(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')^2}}W^a(x,\hat{\mathbf{p}'})\right)\,, \tag{36}$$

from which the color conductivity $\sigma = \omega_p^2/\gamma_g$ follows [1,5,12].

In conclusion, I have derived from the microscopic HTL equations the explicit form of the collision terms recently proposed by Arnold, Son and Yaffe [5] and Blaizot and Iancu [6] which are relevant for Bödeker's effective theory. As the authors of the Ref. [10] have pointed out in a strictly classical context, this approach is simple. It is based on the properties of the plasma fluctuations at equilibrium. In this sense, this approach would be closely connected with linear response theory based on "Kubo" formulas expressing the transport coefficients.

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